Intrusion Prevention through Optimal Stopping

Netcon talk

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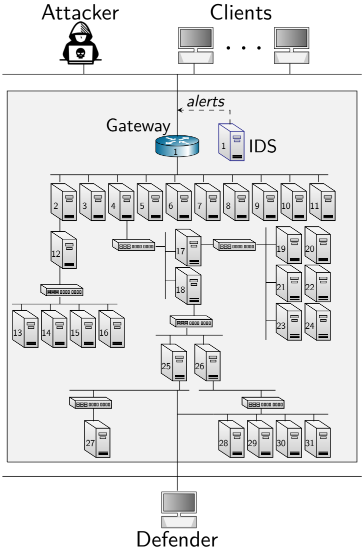
Use Case: Intrusion Prevention

- A **Defender** owns an infrastructure
  - Consists of connected components
  - Components run network services
  - Defender defends the infrastructure by monitoring and active defense
- An **Attacker** seeks to intrude on the infrastructure
  - Has a partial view of the infrastructure
  - Wants to compromise specific components
  - Attacks by reconnaissance, exploitation and pivoting



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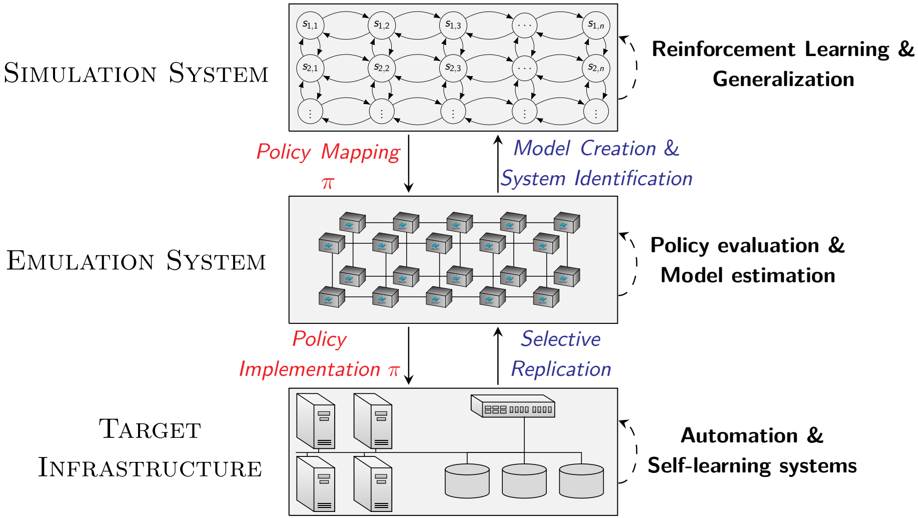

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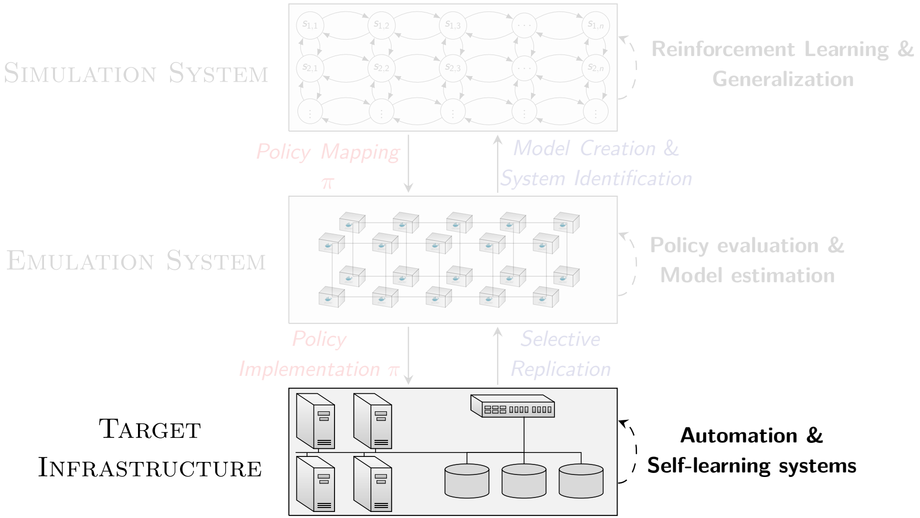

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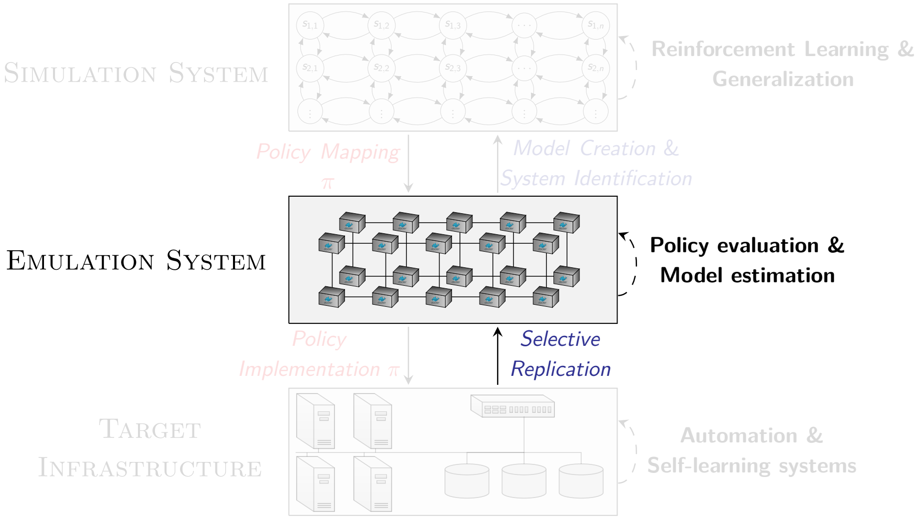

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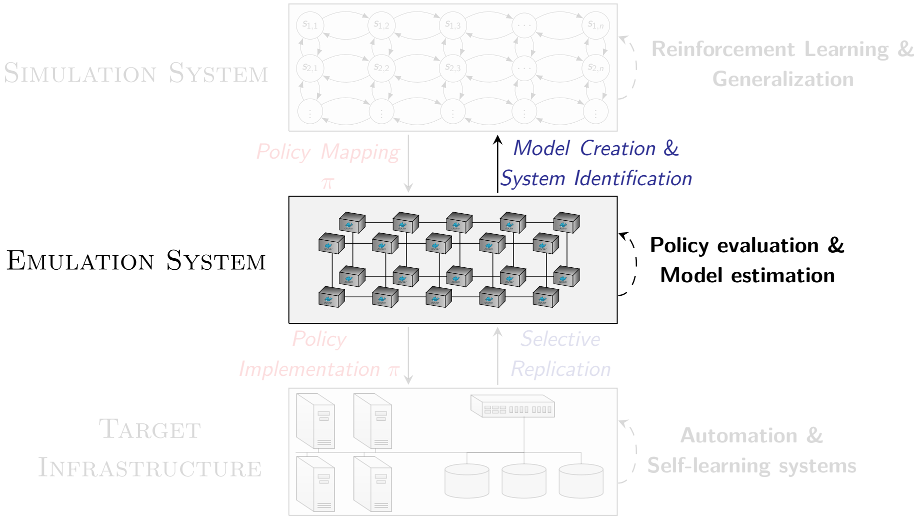

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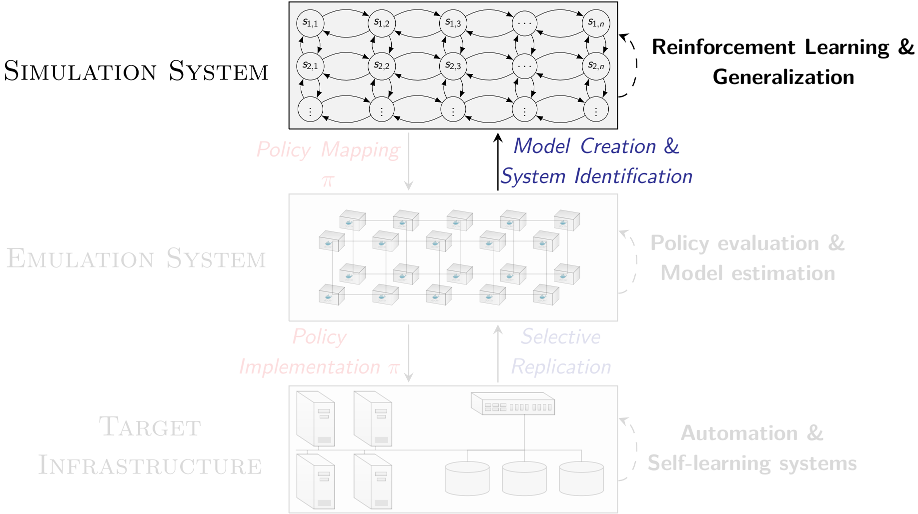

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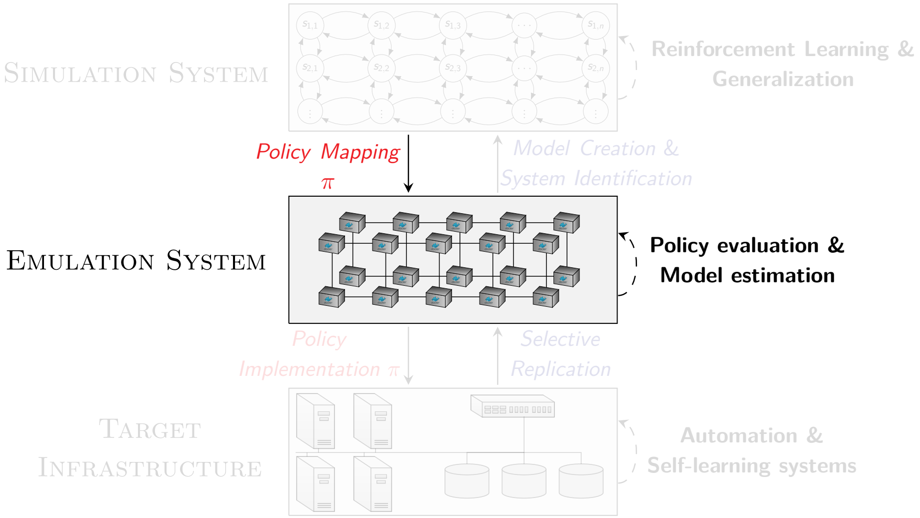

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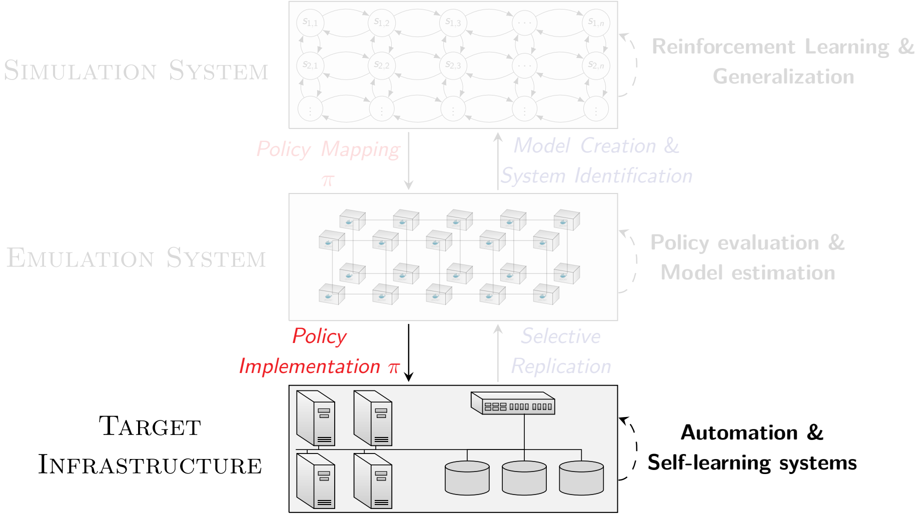

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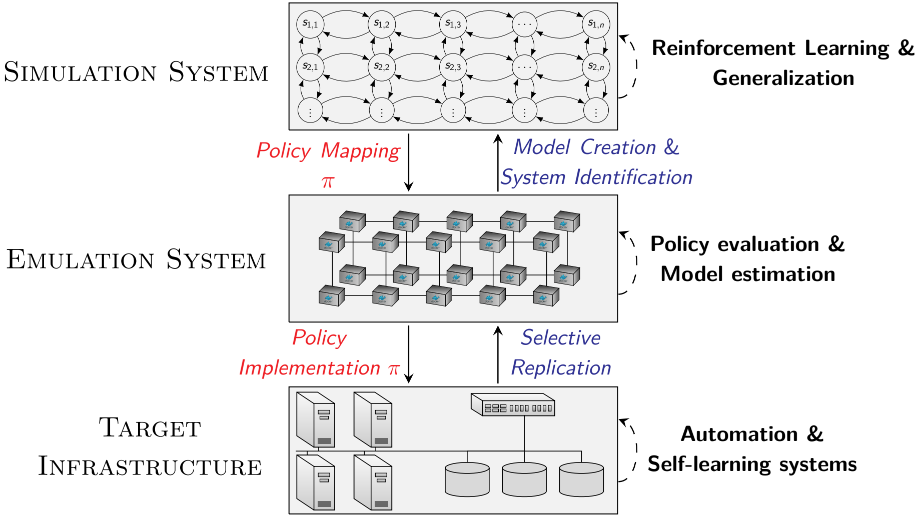

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- **Use Case & Approach:**
  - Intrusion Prevention
  - System identification
  - Reinforcement learning and dynamic programming
- **Formal Model & Background:**
  - Background: POMDPs and optimal stopping
  - Multiple Stopping Problem POMDP
- **Structure of π^***
  - Structural result: Multi-Threshold policy
  - Stopping sets $\mathscr{S}_l$ are connected and nested
  - Conditions for Bayesian filter to be monotone in b
  - Existence of optimal multi-threshold policy π_l^*
- **Conclusion**
  - Numerical evaluation results
  - Conclusion & Future work

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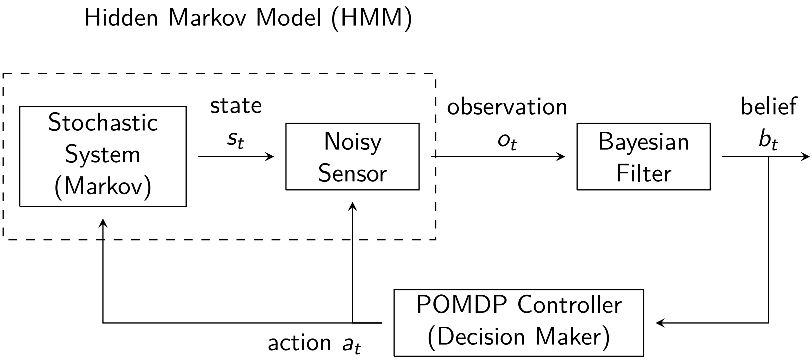
Background: POMDPs



- **POMDP:** $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}^{a_t}_{s_t,s_{t+1}}, \mathcal{R}^{a_t}_{s_t,s_{t+1}}, \gamma, \rho_1, T, \mathcal{O}, \mathcal{Z} \rangle$
- Controlled hidden Markov model, states $s_t \in \mathcal{S}$ are hidden
- Agent observes history $h_t = (\rho_1, a_1, o_1, \ldots, a_{t-1}, o_t) \in \mathcal{H}$

Background: POMDPs

- s_t is Markov: $\mathbb{P}[s_{t+1}|s_t] = \mathbb{P}[s_{t+1}|s_1, \ldots, s_t]$
- $\implies \pi^*(a_t|h_t) = \pi^*(a_t|\mathbb{P}[s_t|h_t]) = \pi^*(a_t|b_t)$
- **Optimality (Bellman) Eq:**

$$\pi^*(b) \in \arg\max_{a \in \mathcal{A}} \left[\sum_s b(s)\mathcal{R}_s^a + \gamma \sum_{o,s,s'} \mathcal{Z}(o,s',a) b(s) \mathcal{P}_{ss'}^a V^*(b_a^o) \right]$$

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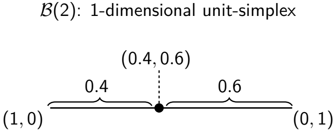

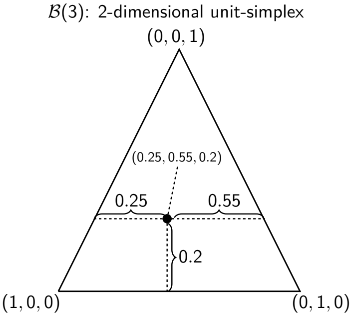

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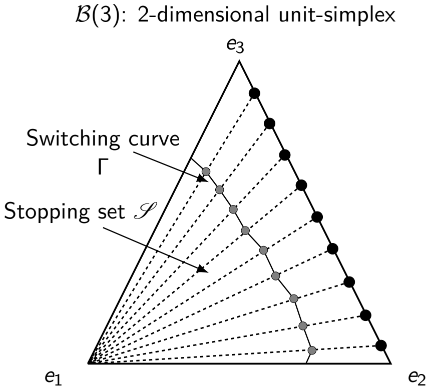

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- $|\mathcal{B}| = \infty$, high-dimensional ($|\mathcal{S}|$) continuous vector
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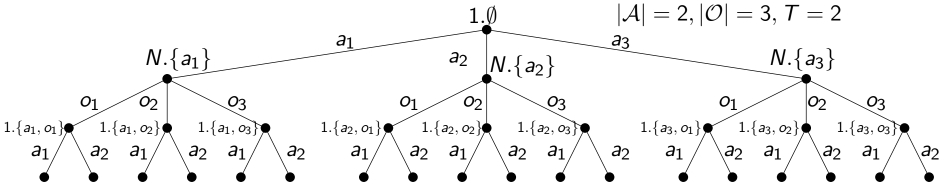

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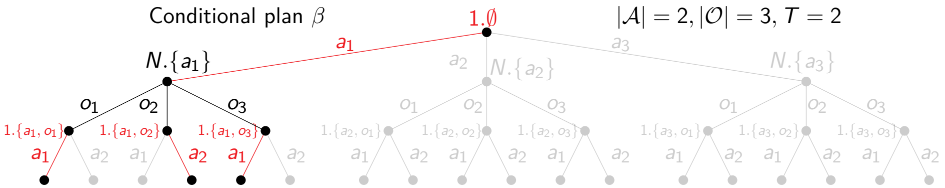

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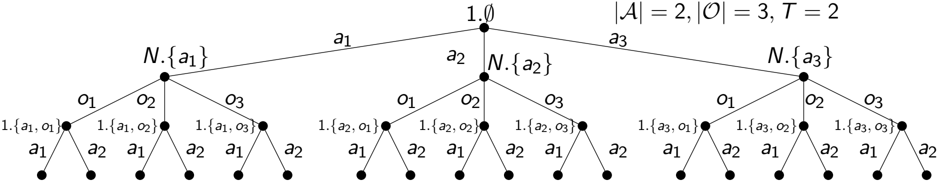

Background: POMDPs

- For each conditional plan $\beta \in \Gamma$:
  - Define vector $\alpha^\beta \in \mathbb{R}^{|\mathcal{S}|}$ such that $\alpha_i^\beta = V^\beta(i)$
  - $\implies V^\beta(b) = b^T \alpha^\beta$ (linear in b).
- Thus, $V^*(b) = \max_{\beta \in \Gamma} b^T \alpha^\beta$ (piece-wise linear and convex[1])

[1] Edward J. Sondik. "The Optimal Control of Partially Observable Markov Processes Over the Infinite Horizon: Discounted Costs". In: *Operations Research* 26.2 (1978), pp. 282–304. ISSN: 0030364X, 15265463. URL: http://www.jstor.org/stable/169635.

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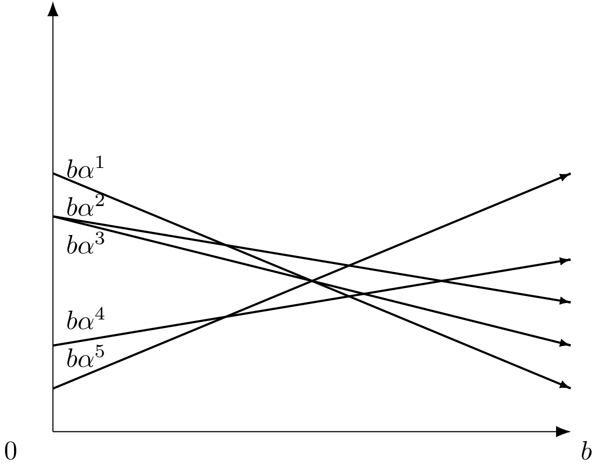

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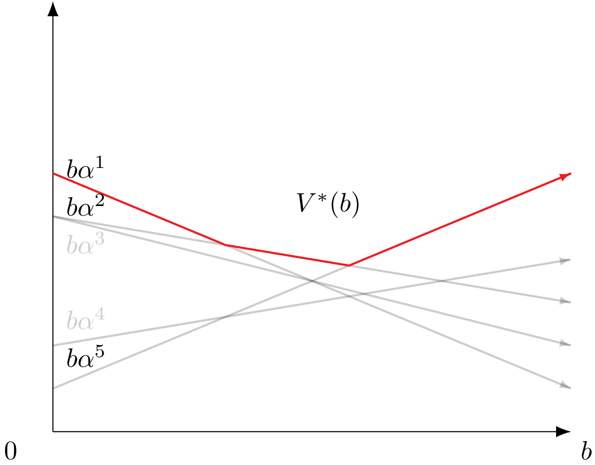
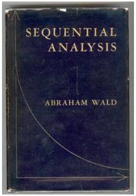


Background: Optimal Stopping

- **History**:
  - Studied in the 18th century to analyze a gambler's fortune
  - Formalized by Abraham Wald in 1947[6]
  - Since then it has been generalized and developed by (Chow[7], Shiryaev & Kolmogorov[8], Bather[9], Bertsekas[10], etc.)



[6] Abraham Wald. *Sequential Analysis*. Wiley and Sons, New York, 1947.

[7] Y. Chow, H. Robbins, and D. Siegmund. "Great expectations: The theory of optimal stopping". In: 1971.

[8] Albert N. Shirayev. *Optimal Stopping Rules*. Reprint of russian edition from 1969. Springer-Verlag Berlin, 2007.

[9] John Bather. *Decision Theory: An Introduction to Dynamic Programming and Sequential Decisions*. USA: John Wiley and Sons, Inc., 2000. ISBN: 0471976490.

[10] Dimitri P. Bertsekas. *Dynamic Programming and Optimal Control*. 3rd. Vol. I. Belmont, MA, USA: Athena Scientific, 2005.

Background: Optimal Stopping

- **The General Problem**:
  - A stochastic process $(s_t)_{t=1}^{T}$ is observed sequentially
  - Two options per t: (i) continue to observe; or (ii) stop
  - Find the *optimal stopping time* τ^*:

$$\tau^* = \arg\max_{\tau} \mathbb{E}_{\tau}\left[\sum_{t=1}^{\tau-1} \gamma^{t-1}\mathcal{R}^{C}_{s_t s_{t+1}} + \gamma^{\tau-1}\mathcal{R}^{S}_{s_\tau s_\tau}\right] \tag{1}$$

  where $\mathcal{R}^{S}_{ss'}$ & $\mathcal{R}^{C}_{ss'}$ are the stop/continue rewards

- **Solution approaches:** the *Markovian approach* and the *martingale approach*.

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- **The General Problem**:
  - A stochastic process $(s_t)_{t=1}^{T}$ is observed sequentially
  - Two options per t: (i) continue to observe; or (ii) stop
  - Find the *optimal stopping time* τ^*:

$$\tau^* = \arg\max_{\tau} \mathbb{E}_\tau \left[\sum_{t=1}^{\tau-1} \gamma^{t-1} \mathcal{R}^{C}_{s_t s_{t+1}} + \gamma^{\tau-1} \mathcal{R}^{S}_{s_\tau s_\tau} \right] \qquad (2)$$

  where $\mathcal{R}^{S}_{ss'}$ & $\mathcal{R}^{C}_{ss'}$ are the stop/continue rewards

- **Solution approaches:** the *Markovian approach* and the *martingale approach*.

Background: Optimal Stopping

- **The Markovian approach**:
  - Model the problem as a MDP or POMDP
  - A policy π^* that satisfies the Bellman-Wald equation is optimal:

$$\pi^*(s) = \underset{\{S,C\}}{\arg\max} \left[\underbrace{\mathbb{E}\left[\mathcal{R}_s^S\right]}_{\text{stop}}, \underbrace{\mathbb{E}\left[\mathcal{R}_s^C + \gamma V^*(s')\right]}_{\text{continue}} \right] \quad \forall s \in \mathcal{S}$$

  - Solve by backward induction, dynamic programming, or reinforcement learning

- **The martingale approach**:
  - Model the state process as an arbitrary stochastic process
  - The reward of the optimal stopping time is given by the *smallest supermartingale that stochastically dominates the process*, called the Snell envelope [13].

Background: Optimal Stopping

- **The Markovian approach**:
  - Assume all rewards are received upon stopping: $R_s^{\emptyset}$
  - $V^*(s)$ **majorizes** $R_s^{\emptyset}$ if $V^*(s) \geq R_s^{\emptyset}\ \forall s \in \mathcal{S}$
  - $V^*(s)$ is **excessive** if $V^*(s) \geq \sum_{s'} \mathcal{P}^{C}_{s's} V^*(s')\ \forall s \in \mathcal{S}$
  - Theorem: $V^*(s)$ is the minimal excessive function which majorizes $R_s^{\emptyset}$.
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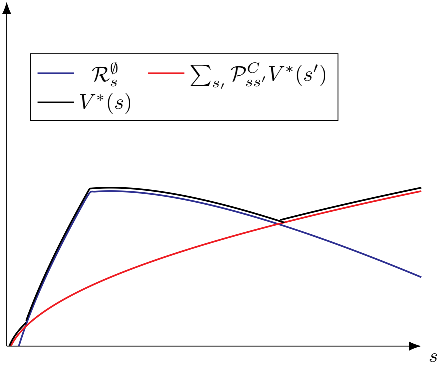

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  - The reward of the optimal stopping time is given by the *smallest supermartingale that stochastically dominates the process*, called the Snell envelope[11].

[11] J. L. Snell. "Applications of martingale system theorems". In: *Transactions of the American Mathematical Society* 73 (1952), pp. 293–312.

Background: Optimal Stopping

- **Applications & Use Cases**:
  - Hypothesis testing[12]
  - Change detection[13],
  - Selling decisions[14],
  - Queue management[15],
  - Industrial control[16],
  - Advertisement scheduling[17], etc.

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[12] Abraham Wald. *Sequential Analysis*. Wiley and Sons, New York, 1947.

[13] Alexander G. Tartakovsky et al. "Detection of intrusions in information systems by sequential change-point methods". In: *Statistical Methodology* 3.3 (2006). ISSN: 1572-3127. DOI: https://doi.org/10.1016/j.stamet.2005.05.003. URL: https://www.sciencedirect.com/science/article/pii/S1572312705000493.

[14] Jacques du Toit and Goran Peskir. "Selling a stock at the ultimate maximum". In: *The Annals of Applied Probability* 19.3 (2009). ISSN: 1050-5164. DOI: 10.1214/08-aap566. URL: http://dx.doi.org/10.1214/08-AAP566.

[15] Arghyadip Roy et al. "Online Reinforcement Learning of Optimal Threshold Policies for Markov Decision Processes". In: *CoRR* (2019). http://arxiv.org/abs/1912.10325. eprint: 1912.10325.

[16] Maben Rabi and Karl H. Johansson. "Event-Triggered Strategies for Industrial Control over Wireless Networks". In: *Proceedings of the 4th Annual International Conference on Wireless Internet*. WICON '08. Maui, Hawaii, USA: ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering), 2008. ISBN: 9789639799363.

[17] Vikram Krishnamurthy, Anup Aprem, and Sujay Bhatt. "Multiple stopping time POMDPs: Structural results & application in interactive advertising on social media". In: *Automatica* 95 (2018), pp. 385–398. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2018.06.013. URL: https://www.sciencedirect.com/science/article/pii/S0005109818303054.

Background: Optimal Stopping

- **Applications & Use Cases**:
  - Hypothesis testing[18]
  - Change detection[19],
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  - Industrial control[22],
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  - **Intrusion prevention**[23] etc.

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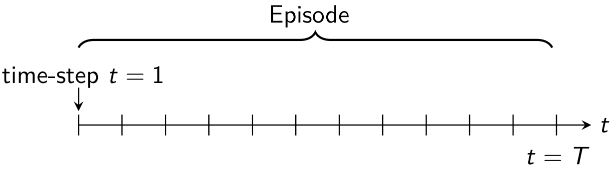
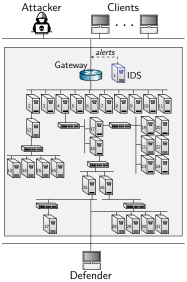
[23] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Formulating Intrusion Prevention as a Stopping Problem





- The system evolves in discrete time-steps.
- Defender observes the infrastructure (IDS, log files, etc.).
- An intrusion occurs at an unknown time.
- The defender can make L stops.
- Each stop is associated with a defensive action
- The final stop shuts down the infrastructure.
- **Based on the observations, when is it optimal to stop?**
- We formalize this problem with a POMDP

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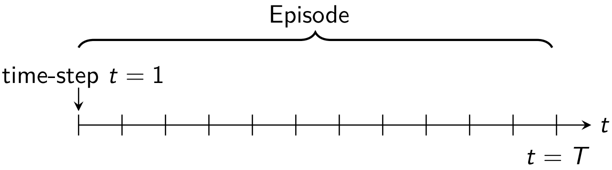

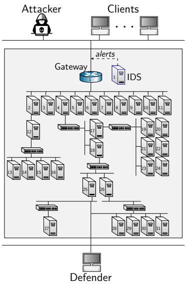

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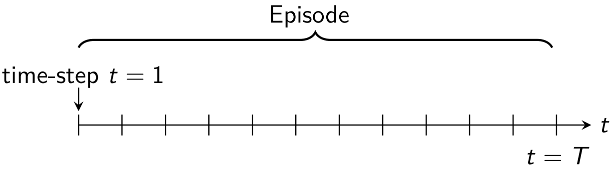

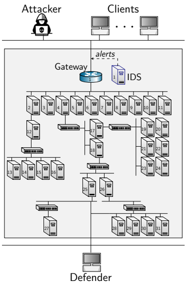

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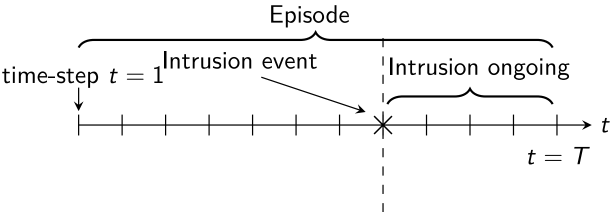

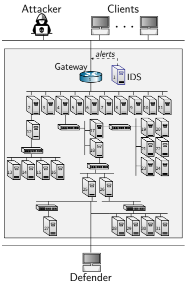

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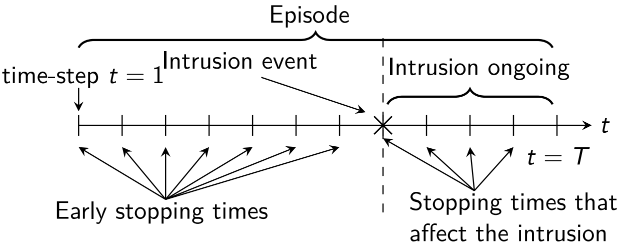

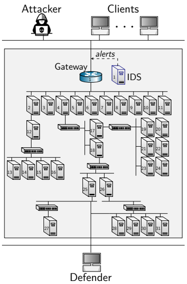

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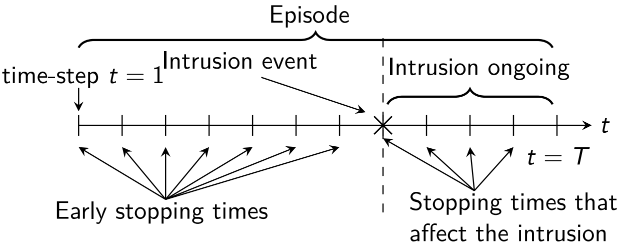

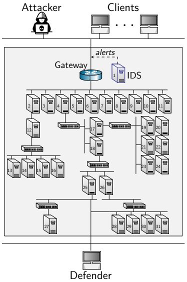

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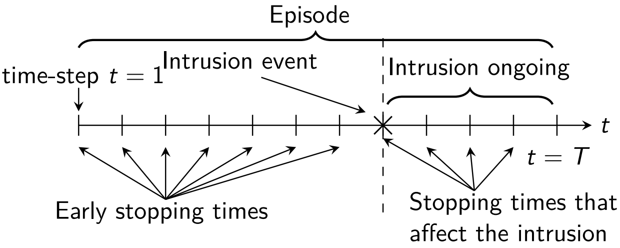

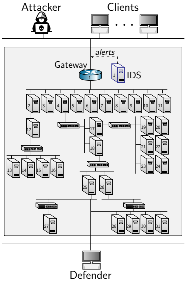

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A Partially Observed MDP Model for the Defender

- **States:**
  - Intrusion state $s_t \in \{0, 1\}$, terminal $\emptyset$.
- **Observations:**
  - Severe/Warning IDS Alerts $(\Delta x, \Delta y)$, Login attempts Δz, stops remaining $l_t \in \{1, .., L\}$, $f_{XYZ}(\Delta x, \Delta y, \Delta z | s_t)$
- **Actions:**
  - "Stop" (S) and "Continue" (C)
- **Rewards:**
  - Reward: security and service. Penalty: false alarms and intrusions
- **Transition probabilities:**
  - Bernoulli process $(Q_t)_{t=1}^{T} \sim Ber(p)$ defines intrusion start $I_t \sim Ge(p)$
- **Objective and Horizon:**
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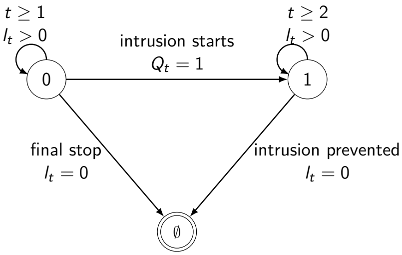

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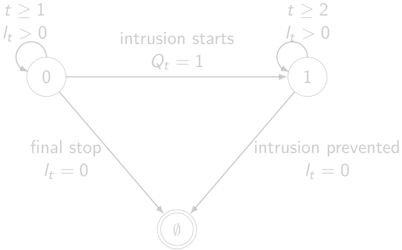

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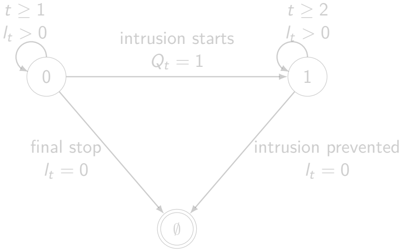

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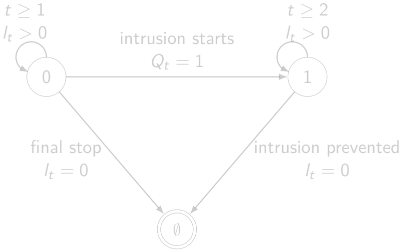

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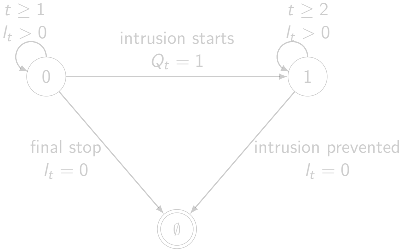

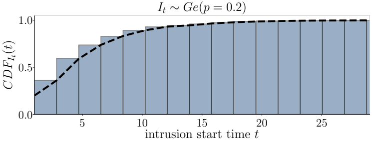

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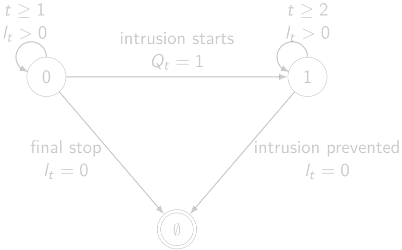

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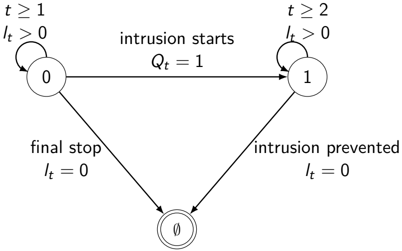

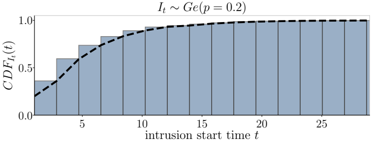

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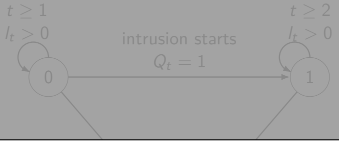

We analyze the structure of π^* using POMDP & stopping theory

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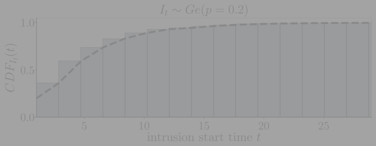

Outline

- **Use Case & Approach:**
  - Intrusion Prevention
  - System identification
  - Reinforcement learning and dynamic programming
- **Formal Model & Background:**
  - Background: POMDPs and optimal stopping
  - Multiple Stopping Problem POMDP
- **Structure of π^***
  - Structural result: Multi-Threshold policy
  - Stopping sets $\mathscr{S}_l$ are connected and nested
  - Conditions for Bayesian filter to be monotone in b
  - Existence of optimal multi-threshold policy π^*_l
- **Conclusion**
  - Numerical evaluation results
  - Conclusion & Future work

Structural Result: Optimal Multi-Threshold Policy

Theorem

Given the intrusion prevention POMDP, the following holds:

1. $\mathscr{S}_{l-1} \subseteq \mathscr{S}_l$ *for* $l = 2, \ldots L$.
2. *If* $L = 1$*, there exists an optimal threshold* $\alpha^* \in [0,1]$ *and an optimal policy of the form:*

$$\pi_L^*(b(1)) = S \iff b(1) \geq \alpha^* \tag{3}$$

3. *If* $L \geq 1$ *and* f_{XYZ} *is totally positive of order 2 (TP2), there exists* L *optimal thresholds* $\alpha_l^* \in [0,1]$ *and an optimal policy of the form:*

$$\pi_l^*(b(1)) = S \iff b(1) \geq \alpha_l^*, \qquad l = 1, \ldots, L \tag{4}$$

where α_l^* *is decreasing in* l.

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3. *If* $L \geq 1$ *and* f_{XYZ} *is totally positive of order 2 (TP2), there exists* L *optimal thresholds* $\alpha^*_l \in [0, 1]$ *and an optimal policy of the form:*

$$\pi^*_l(b(1)) = S \iff b(1) \geq \alpha^*_l, \qquad l = 1, \ldots, L \tag{6}$$

where α^*_l *is decreasing in* l.

Structural Result: Optimal Multi-Threshold Policy

Theorem

Given the intrusion prevention POMDP, the following holds:

1. $\mathscr{S}_{l-1} \subseteq \mathscr{S}_l$ *for* $l = 2, \ldots L$.
2. *If* $L = 1$, *there exists an optimal threshold* $\alpha^* \in [0, 1]$ *and an optimal policy of the form:*

$$\pi_L^*(b(1)) = S \iff b(1) \geq \alpha^* \tag{7}$$

3. *If* $L \geq 1$ *and* f_{XYZ} *is totally positive of order 2 (TP2), there exists* L *optimal thresholds* $\alpha_l^* \in [0, 1]$ *and an optimal policy of the form:*

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3. *If* $L \geq 1$ *and* f_{XYZ} *is totally positive of order 2 (TP2), there exists* L *optimal thresholds* $\alpha_l^* \in [0, 1]$ *and an optimal policy of the form:*

$$\pi_l^*(b(1)) = S \iff b(1) \geq \alpha_l^*, \qquad l = 1, \ldots, L \tag{10}$$

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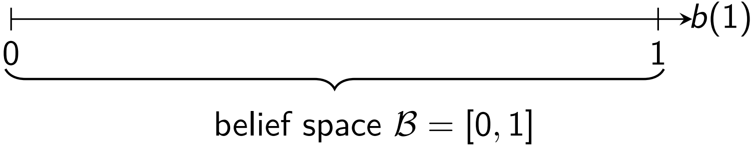
Structural Result: Optimal Multi-Threshold Policy



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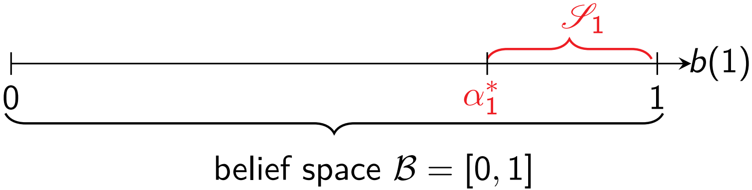

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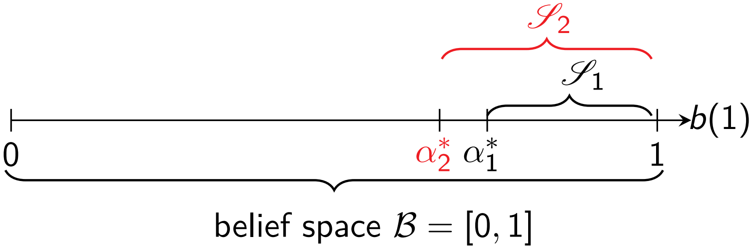

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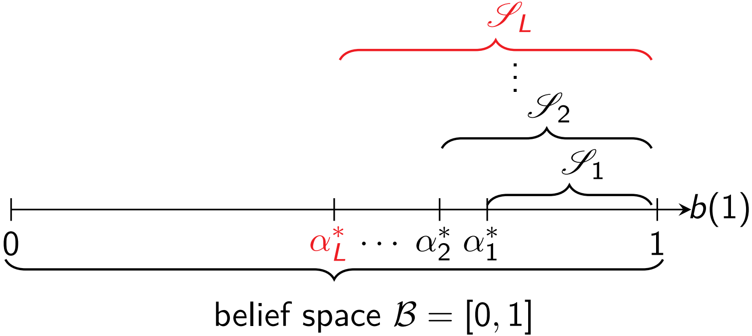

Proofs: $\mathscr{S}_1$ is convex[24]

- $\mathscr{S}_1$ is convex if:
  - for any two belief states $b_1, b_2 \in \mathscr{S}_1$
  - any convex combination of b_1, b_2 is also in $\mathscr{S}_1$
  - i.e. $b_1, b_2 \in \mathscr{S}_1 \implies \lambda b_1 + (1-\lambda)b_2 \in \mathscr{S}_1$ for $\lambda \in [0,1]$.

- Since $V^*(b)$ is convex:

$$V^*(\lambda b_1 + (1-\lambda)b_2) \leq \lambda V^*(b_1) + (1-\lambda)V(b_2)$$

- Since $b_1, b_2 \in \mathscr{S}_1$:

$$V^*(b_1) = Q^*(b_1, S) \qquad S{=}\text{stop}$$
$$V^*(b_2) = Q^*(b_2, S) \qquad S{=}\text{stop}$$

[24] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

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- Since $b_1, b_2 \in \mathscr{S}_1$:

$$\begin{aligned} V^*(b_1) &= Q^*(b_1, S) && S\text{=stop} \\ V^*(b_2) &= Q^*(b_2, S) && S\text{=stop} \end{aligned}$$

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Proofs: $\mathscr{S}_1$ is convex[27]

Proof.

Assume $b_1, b_2 \in \mathscr{S}_1$. Then for any $\lambda \in [0,1]$:

$$
\begin{aligned}
V^*(\lambda b_1(1) + (1-\lambda) b_2(1)) &\leq \lambda V^*(b_1(1)) + (1-\lambda) V^*(b_2(1)) \\
&= \lambda Q^*(b_1, S) + (1-\lambda) Q^*(b_2, S)
\end{aligned}
$$

□

[27] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: $\mathscr{S}_1$ is convex[28]

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Assume $b_1, b_2 \in \mathscr{S}_1$. Then for any $\lambda \in [0,1]$:

$$\begin{aligned}
V^*(\lambda b_1(1) + (1-\lambda) b_2(1)) &\leq \lambda V^*(b_1(1)) + (1-\lambda) V^*(b_2(1)) \\
&= \lambda Q^*(b_1, S) + (1-\lambda) Q^*(b_2, S) \\
&= \lambda \mathcal{R}^{\emptyset}_{b_1} + (1-\lambda)\mathcal{R}^{\emptyset}_{b_2} \\
&= \sum_s (\lambda b_1(s) + (1-\lambda) b_2(s)) \mathcal{R}^{\emptyset}_s
\end{aligned}$$

□

[28] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: $\mathscr{S}_1$ is convex[29]

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Assume $b_1, b_2 \in \mathscr{S}_1$. Then for any $\lambda \in [0,1]$:

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V^*(\lambda b_1(1) + (1-\lambda) b_2(1)) &\leq \lambda V^*(b_1(1)) + (1-\lambda) V^*(b_2(1)) \\
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&= \sum_s (\lambda b_1(s) + (1-\lambda) b_2(s)) \mathcal{R}^{\emptyset}_s \\
&= Q^*(\lambda b_1 + (1-\lambda) b_2, S)
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[29] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: $\mathscr{S}_1$ is convex[30]

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&= \lambda \mathcal{R}^{\emptyset}_{b_1} + (1-\lambda)\mathcal{R}^{\emptyset}_{b_2} \\
&= \sum_s (\lambda b_1(s) + (1-\lambda) b_2(s)) \mathcal{R}^{\emptyset}_s \\
&= Q^*(\lambda b_1 + (1-\lambda) b_2, S) \\
&\leq V^*(\lambda b_1(1) + (1-\lambda) b_2(1))
\end{aligned}$$

the last inequality is because V^* is optimal. The second-to-last is because there is just a single stop. □

[30] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: $\mathscr{S}_1$ is convex[31]

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Assume $b_1, b_2 \in \mathscr{S}_1$. Then for any $\lambda \in [0,1]$:

$$\begin{aligned} V^*(\lambda b_1(1) + (1-\lambda) b_2(1)) &\leq \lambda V^*(b_1(1)) + (1-\lambda) V^*(b_2(1)) \\ &= \lambda Q^*(b_1, S) + (1-\lambda) Q^*(b_2, S) \\ &= Q^*(\lambda b_1 + (1-\lambda) b_2, S) \\ &\leq V^*(\lambda b_1(1) + (1-\lambda) b_2(1)) \end{aligned}$$

the last inequality is because V^* is optimal. The second-to-last is because there is just a single stop. Hence:

$$Q^*(\lambda b_1 + (1-\lambda) b_2, S) = V^*(\lambda b_1(1) + (1-\lambda) b_2(1))$$

$b_1, b_2 \in \mathscr{S}_1 \implies (\lambda b_1 + (1-\lambda)) \in \mathscr{S}_1$. Therefore $\mathscr{S}_1$ is convex. $\square$

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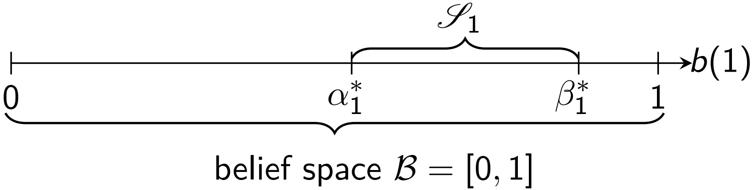

[32] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: Single-threshold policy is optimal if $L = 1$[33]

- In our case, $\mathcal{B} = [0,1]$. We know $\mathscr{S}_1$ is a convex subset of $\mathcal{B}$.
- Consequence, $\mathscr{S}_1 = [\alpha^*, \beta^*]$. We show that $\beta^* = 1$.
- If $b(1) = 1$, using our definition of the reward function, the Bellman equation states:

$$\pi^*(1) \in \arg\max_{\{S,C\}} \Bigg[\underbrace{150 + V^*(\emptyset)}_{a=S}, \underbrace{-90 + \sum_{o \in \mathcal{O}} \mathcal{Z}(o,1,C) V^*(b_C^o(1))}_{a=C} \Bigg]$$

$$= \arg\max_{\{S,C\}} \Bigg[\underbrace{150}_{a=S}, \underbrace{-90 + V^*(1)}_{a=C} \Bigg] = S \quad \text{i.e } \pi^*(1) = \text{Stop}$$

- Hence $1 \in \mathscr{S}_1$. It follows that $\mathscr{S}_1 = [\alpha^*, 1]$ and:

$$\pi^*(b(1)) = \begin{cases} S & \text{if } b(1) \geq \alpha^* \\ C & \text{otherwise} \end{cases}$$

[33] Kim Hammar and Rolf Stadler. "Learning Intrusion Prevention Policies through Optimal Stopping". In: *International Conference on Network and Service Management (CNSM 2021)*. https://arxiv.org/pdf/2106.07160.pdf. Izmir, Turkey, 2021.

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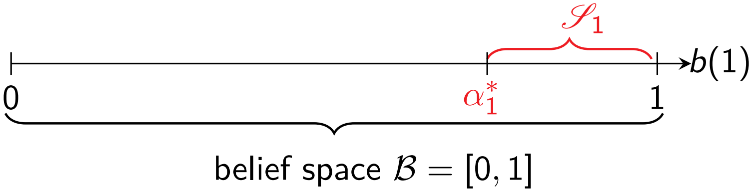

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[36]

- If $b(1) \in \mathscr{S}_{l-1}$, we use the Bellman eq. to obtain:
- $$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} \geq \sum_o \mathbb{P}^o_{b(1)}\Big(V^*_{l-1}(b^o(1)) - V^*_{l-2}(b^o(1))\Big)$$
- **We show that LHS is non-decreasing in l and RHS is non-increasing in l.**
- LHS is non-decreasing by definition of reward function.
- We show that RHS is non-increasing by induction on $k = 0, 1 \ldots$ where k is the iteration of value iteration.
- We know $\lim_{k\to\infty} V^k(b) = V^*(b)$.
- Define $W^k_l(b(1)) = V^k_l(b(1)) - V^k_{l-1}(b(1))$

[36] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

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Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[40]

Proof.

$W_l^0(b(1)) = 0 \ \forall l$. Assume $W_{l-1}^{k-1}(b(1)) - W_l^{k-1}(b(1)) \geq 0$. $\square$

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$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = 2V_{l-1}^k - V_{l-2}^k - V_l^k$$

□

[41] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

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$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = 2V_{l-1}^k - V_{l-2}^k - V_l^k = 2\mathcal{R}_{b(1)}^{a_{l-1}^k} - \mathcal{R}_{b(1)}^{a_l^k} - \mathcal{R}_{b(1)}^{a_{l-2}^k}$$
$$+ \sum_{o\in\mathcal{O}} \mathbb{P}_{b(1)}^o \Big(2V_{l-1-a_{l-1}^k}^{k-1}(b(1)) - V_{l-a_l^k}^{k-1}(b(1)) - V_{l-2-a_{l-2}^k}^{k-1}(b(1))\Big)$$

Want to show that the above is non-negative. This depends on $a_l^k, a_{l-1}^k, a_{l-2}^k$. $\square$

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Want to show that the above is non-negative. This depends on $a_l^k, a_{l-1}^k, a_{l-2}^k$.

There are four cases to consider:

1. $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{S}_{l-2}^k$
2. $b(1) \in \mathscr{S}_l^k \cap \mathscr{C}_{l-1}^k \cap \mathscr{C}_{l-2}^k$
3. $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{C}_{l-2}^k$
4. $b(1) \in \mathscr{C}_l^k \cap \mathscr{C}_{l-1}^k \cap \mathscr{C}_{l-2}^k$

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[44]

Proof.

If $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{S}_{l-2}^k$, then:

$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-2}^{k-1}(b^o(1)) - W_{l-1}^{k-1}(b^o(1)) \Big)$$

which is non-negative by the induction hypothesis. $\square$

[44] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[45]

Proof.

If $b(1) \in \mathscr{S}^k_l \cap \mathscr{S}^k_{l-1} \cap \mathscr{S}^k_{l-2}$, then:

$$W^k_{l-1}(b(1)) - W^k_l(b(1)) = \sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(W^{k-1}_{l-2}(b^o(1)) - W^{k-1}_{l-1}(b^o(1)) \Big)$$

which is non-negative by the induction hypothesis.

If $b(1) \in \mathscr{S}^k_l \cap \mathscr{C}^k_{l-1} \cap \mathscr{C}^k_{l-2}$, then:

$$W^k_l(b(1)) - W^k_{l-1}(b(1)) = \mathcal{R}^C_{b(1)} - \mathcal{R}^S_{b(1)} + \sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(W^{k-1}_{l-1}(b^o(1)) \Big)$$

$\square$

[45] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[46]

Proof.

If $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{S}_{l-2}^k$, then:

$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-2}^{k-1}(b^o(1)) - W_{l-1}^{k-1}(b^o(1)) \Big)$$

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If $b(1) \in \mathscr{S}_l^k \cap \mathscr{C}_{l-1}^k \cap \mathscr{C}_{l-2}^k$, then:

$$W_l^k(b(1)) - W_{l-1}^k(b(1)) = \mathcal{R}_{b(1)}^C - \mathcal{R}_{b(1)}^S + \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-1}^{k-1}(b^o(1)) \Big)$$

Bellman eq. implies, if $b(1) \in \mathscr{C}_{l-1}$, then:

$$\mathcal{R}_{b(1)}^C - \mathcal{R}_{b(1)}^S + \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-1}^{k-1}(b^o(1)) \Big) \geq 0$$

$\square$

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[47]

Proof.

If $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{C}_{l-2}^k$, then:

$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = \mathcal{R}_{b(1)}^S - \mathcal{R}_{b(1)}^C - \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-1}^{k-1}(b^o(1)) \Big)$$

$\square$

[47] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$ [48]

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[48] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[49]

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If $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k \cap \mathscr{C}_{l-2}^k$, then:

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If $b(1) \in \mathscr{C}_l^k \cap \mathscr{C}_{l-1}^k \cap \mathscr{C}_{l-2}^k$, then:

$$W_{l-1}^k(b(1)) - W_l^k(b(1)) = \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(W_{l-1}^{k-1}(b^o(1)) - W_l^{k-1}(b^o(1)) \Big)$$

which is non-negative by the induction hypothesis. □

[49] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[50]

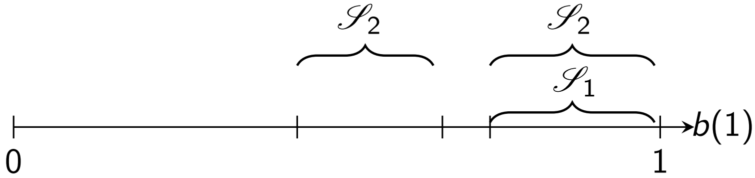
Hence, we have shown that W_l^k is non-increasing in l.

It follows that $b(1) \in \mathscr{S}_{l-1} \implies b(1) \in \mathscr{S}_l$.

[50] T. Nakai. "The problem of optimal stopping in a partially observable Markov chain". In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Nested stopping sets $\mathscr{S}_l \subseteq \mathscr{S}_{1+l}$[51]

$\mathscr{S}_1 \subseteq \mathscr{S}_2$ still allows:



[51] T. Nakai. “The problem of optimal stopping in a partially observable Markov chain”. In: *Journal of Optimization Theory and Applications* 45.3 (1985), pp. 425–442. ISSN: 1573-2878. DOI: 10.1007/BF00938445. URL: https://doi.org/10.1007/BF00938445.

Proofs: Necessary Condition, Total Positivity of Order 2[52]

- A row-stochastic matrix is totally positive of order 2 (TP2) if:
  - The rows of the matrix are stochastically monotone
  - Equivalently, all second-order minors are non-negative.
- Example:

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \tag{11}$$

There are $\binom{3}{2}^2$ second-order minors:

$$det \begin{bmatrix} 0.3 & 0.5 \\ 0.2 & 0.4 \end{bmatrix} = 0.02, \quad det \begin{bmatrix} 0.2 & 0.4 \\ 0.1 & 0.2 \end{bmatrix} = 0, \ldots\text{etc.} \tag{12}$$

Since all minors are non-negative, the matrix is TP2

[52] Samuel Karlin. "Total positivity, absorption probabilities and applications". In: *Transactions of the American Mathematical Society* 111 (1964).

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Since all minors are non-negative, the matrix is TP2

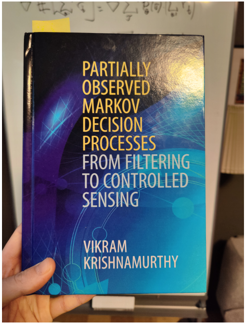
[53] Samuel Karlin. "Total positivity, absorption probabilities and applications". In: *Transactions of the American Mathematical Society* 111 (1964).

Proofs: Monotone belief update[55]

Theorem (Monotone belief update)

Given two beliefs $b_1(1) \geq b_2(1)$, if the transition probabilities and the observation probabilities are TP2, then $b^o_{a,1}(1) \geq b^o_{a,2}(1)$, where $b^o_{a,1}(1)$ and $b^o_{a,2}(1)$ denote the beliefs updated with the Bayesian filter after taking action $a \in \mathcal{A}$ and observing $o \in \mathcal{O}$.

See Theorem 10.3.1 and proof on pp 225,238 in[54]



[54] Vikram Krishnamurthy. *Partially Observed Markov Decision Processes: From Filtering to Controlled Sensing*. Cambridge University Press, 2016. DOI: 10.1017/CBO9781316471104.

Proofs: Connected stopping sets $\mathscr{S}_l$[56]

- $\mathscr{S}_l$ is connected if $b(1) \in \mathscr{S}_l, b'(1) \geq b(1) \implies b'(1) \in \mathscr{S}_l$
- If $b(1) \in \mathscr{S}_l$ we use the Bellman eq. to obtain:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + \sum_o \mathbb{P}^o_{b(1)}\Big(V^*_{l-1}(b^o(1)) - V^*_l(b^o(1))\Big) \geq 0$$

- The inequality above should also hold for any $b'(1) \geq b(1)$
- Transition probabilities are TP2 by definition
- We assume observation probabilities are TP2
- It follows that the belief updates are monotone

- Hence, it is sufficient to show that:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^*_{l-1}(b(1)) - V^*_l(b(1))$$

  is weakly increasing in $b(1)$.
- We prove this by induction on k.

[56] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

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[59] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

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- $\mathscr{S}_l$ is connected if $b(1) \in \mathscr{S}_l, b'(1) \geq b(1) \implies b'(1) \in \mathscr{S}_l$
- If $b(1) \in \mathscr{S}_l$ we use the Bellman eq. to obtain:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + \sum_o \mathbb{P}^o_{b(1)}\Big(V^*_{l-1}(b^o(1)) - V^*_l(b^o(1))\Big) \geq 0$$

- The inequality above should also hold for any $b'(1) \geq b(1)$
- Transition probabilities are TP2 by definition
- We assume observation probabilities are TP2
- It follows that the belief updates are monotone

- Hence, it is sufficient to show that:

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- We prove this by induction on k.

[60] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[61]

Assume $\mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} + V^{k-1}_{l-1}(b(1)) - V^{k-1}_{l}(b(1))$ is weakly increasing in $b(1)$.

$$\mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} + V^{k}_{l-1}(b(1)) - V^{k}_{l}(b(1)) = \mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} +$$
$$\mathcal{R}^{a^k_{l-1}}_{b(1)} - \mathcal{R}^{a^k_{l}}_{b(1)} + \sum_{o\in\mathcal{O}} \mathbb{P}^{o}_{b(1)}\Big(V^{k-1}_{l-1-a^k_{l-1}}(b^o(1)) - V^{k-1}_{l-a^k_l}(b^o(1))\Big)$$

[61] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[62]

Assume $\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^{k-1}_{l-1}(b(1)) - V^{k-1}_{l}(b(1))$ is weakly increasing in $b(1)$.

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^{k}_{l-1}(b(1)) - V^{k}_{l}(b(1)) = \mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + \mathcal{R}^{a^k_{l-1}}_{b(1)} - \mathcal{R}^{a^k_{l}}_{b(1)} + \sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(V^{k-1}_{l-1-a^k_{l-1}}(b^o(1)) - V^{k-1}_{l-a^k_l}(b^o(1)) \Big)$$

Want to show that the above is weakly-increasing in $b(1)$. This depends on a^k_l and a^k_{l-1}.

[62] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[63]

Assume $\mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} + V^{k-1}_{l-1}(b(1)) - V^{k-1}_{l}(b(1))$ is weakly increasing in $b(1)$.

$$\mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} + V^{k}_{l-1}(b(1)) - V^{k}_{l}(b(1)) = \mathcal{R}^{S}_{b(1)} - \mathcal{R}^{C}_{b(1)} +$$

$$\mathcal{R}^{a^k_{l-1}}_{b(1)} - \mathcal{R}^{a^k_{l}}_{b(1)} + \sum_{o \in \mathcal{O}} \mathbb{P}^{o}_{b(1)} \Big(V^{k-1}_{l-1-a^k_{l-1}}(b^o(1)) - V^{k-1}_{l-a^k_l}(b^o(1)) \Big)$$

Want to show that the above is weakly-increasing in $b(1)$. This depends on a^k_l and a^k_{l-1}.

There are three cases to consider:

1. $b(1) \in \mathscr{S}^k_l \cap \mathscr{S}^k_{l-1}$
2. $b(1) \in \mathscr{S}^k_l \cap \mathscr{C}^k_{l-1}$
3. $b(1) \in \mathscr{C}^k_l \cap \mathscr{C}^k_{l-1}$

[63] Kim Hammar and Rolf Stadler. “Intrusion Prevention through Optimal Stopping”. In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[64]

Proof.

If $b(1) \in \mathscr{S}_l \cap \mathscr{S}_{l-1}$, then:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^k_{l-1}(b(1)) - V^k_l(b(1)) =$$

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} \sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(V^{k-1}_{l-2}(b^o(1)) - V^{k-1}_{l-1}(b^o(1)) \Big)$$

which is weakly increasing in $b(1)$ by the induction hypothesis. $\square$

[64] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[65]

Proof.

If $b(1) \in \mathscr{S}_l^k \cap \mathscr{S}_{l-1}^k$, then:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^k_{l-1}(b(1)) - V^k_l(b(1)) =$$
$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} \sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(V^{k-1}_{l-2}(b^o(1)) - V^{k-1}_{l-1}(b^o(1)) \Big)$$

which is weakly increasing in $b(1)$ by the induction hypothesis.

If $b(1) \in \mathscr{S}_l^k \cap \mathscr{C}_{l-1}^k$, then:

$$\mathcal{R}^S_{b(1)} - \mathcal{R}^C_{b(1)} + V^k_{l-1}(b(1)) - V^k_l(b(1)) =$$
$$\sum_{o \in \mathcal{O}} \mathbb{P}^o_{b(1)} \Big(V^{k-1}_{l-1}(b^o(1)) - V^{k-1}_{l-1}(b^o(1)) \Big) = 0$$

which is trivially weakly increasing in $b(1)$. $\square$

[65] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Connected stopping sets $\mathscr{S}_l$[66]

Proof.

If $b(1) \in \mathscr{C}_l^k \cap \mathscr{C}_{l-1}^k$, then:

$$\mathcal{R}_{b(1)}^S - \mathcal{R}_{b(1)}^C + V_{l-1}^k(b(1)) - V_l^k(b(1)) =$$

$$\mathcal{R}_{b(1)}^S - \mathcal{R}_{b(1)}^C \sum_{o \in \mathcal{O}} \mathbb{P}_{b(1)}^o \Big(V_{l-1}^{k-1}(b^o(1)) - V_l^{k-1}(b^o(1)) \Big)$$

which is weakly increasing in $b(1)$ by the induction hypothesis. □

Hence, if $b(1) \in \mathscr{S}_l$ and $b'(1) \geq b(1)$ then $b'(1) \in \mathscr{S}_l$.
Therefore, $\mathscr{S}_l$ is connected.

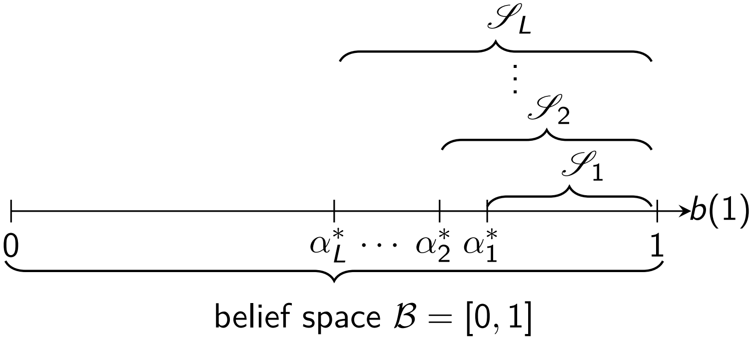
[66] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Proofs: Optimal multi-threshold policy π_l^*[67]

We have shown that:

- $\mathscr{S}_1 = [\alpha_1^*, 1]$
- $\mathscr{S}_l \subseteq \mathscr{S}_{l+1}$
- $\mathscr{S}_l$ is connected (convex) for $l = 1, \ldots, L$

It follows that, $\mathscr{S}_l = [\alpha_l^*, 1]$ and $\alpha_1^* \geq \alpha_2^* \geq \ldots \geq \alpha_L^*$.



[67] Kim Hammar and Rolf Stadler. "Intrusion Prevention through Optimal Stopping". In: (). 2021, https://arxiv.org/abs/2111.00289. arXiv: 2111.00289.

Conclusions & Future Work

- **Conclusions:**
  - We develop a *method* to automatically learn security policies
    - (1) emulation system; (2) system identification; (3) simulation system; (4) reinforcement learning and (5) domain randomization and generalization.
  - We apply the method to an **intrusion prevention use case**
  - We formulate intrusion prevention as a multiple stopping problem
    - We present a POMDP model of the use case
    - We apply the stopping theory to establish structural results of the optimal policy
    - We show numerical results in realistic emulation environment (not included in this presentation)
- **Our research plans:**
  - Extending the model
    - Active attacker: Partially Observed Stochastic Game, Equilibrium analysis
    - Less restrictions on defender

    Scaling up the emulation system:
    - More realistic traffic emulation
    - Non-static infrastructures